FUTUREMATH ERASMUS+ PROJECT

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UNIVERZITET METROPOLITAN BEOGRAD

CS115 - DISKRETNE STRUKTURE

RECURRENCE RELATIONS

Lekcija 17

PRIRUČNIK ZA STUDENTE

CS115 - DISKRETNE STRUKTURE

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RECURRENCE RELATIONS

Uvod

INTRODUCTION

The focus of this lesson is on recurrsion, with special reference to recurrent relations

Methods for solving some recurrent relations, that is, finding an explicit expression for a general member of a recursively determined sequence in special cases
- linear recursion with constant coefficients
- homogeneous linear recursion with constant second-order coefficients

Poglavlje 1

RECURRENCE RELATION

DEFINITION OF RECURRSION

Recursion is a problem-solving method that involves breaking down problems into smaller subproblems

Recurrsion is a method of problem solving that involves breaking down a problem into smaller subproblems, until a sufficiently smallsubproblems.

Recurrsive functions are functions that call themselves. They make it easy and understandable to solve problems that are recursive in nature.

Problems that can be solved through recurrsion are called recurrsive (recurrent) problem. A large number of software solutions are a recursive problem.

We have previously studied recurrent functions: the factorial, the Fibonacci sequence, and the Ackerman function. Here we discuss some types of recursively defined arrays (a_n) and their explicit representations. Recall that a series of functions is defined over a domain

N = {1, 2, . . . }

or

N_0 = N ∪ {0}

Example

Consider a sequence beginning with the number 3, and each subsequent element is obtained by multiplying the previous element by 2, that is, sequence
3, 6, 12, 24, 48, ...

This sequence can be recursive in the following way
$a_0 = 3$, $a_k = 2a_{k-1}$ for $k \geq 1$,
or
$a_0 = 3$, $a_{k+1} = 2a_k$ for $k \geq 0$.

Obviously, the general member of this sequence is

$a_k = 3 - 2_k$

EXAMPLES OF RECURRSION IN PROGRAMMING

Problems that can be solved through recursion are called recursive (recurrent) problems

Examples

(application in programming)

One of the elementary programming problems issequnce sorting.

Consider a growing sequence of n members and let its n-1 be members

1,2,3…n-1

By adding the n-th member, we get asequence of n members, so we will find the place where the n-th member should be placed.

The sorting problem is reduced to finding a place in an already sorted array. The problem of sorting asequnce of n members is reduced to sorting a sequnce of n-1 members.

We apply the same procedure to this sequence recurrsively and in the end we will come to asequence of records with length 1.

Examples
(application in mathematics)
A mathematical example of recursion is the definition of factorials. Factorial of natural number n, in the notation n! can be defined as
$n! = 1 * 2 \ldots * (n - 1) * n$
An alternative way to define the factorial is:
$0! = 1$
$n! = (n - 1)! * n$
where the first statement defines the simplest case, and the second statement defines the reduction of the problem of order n to the simpler problem of order n-1.
That way, no matter how big n was after finally many steps we will come to a trivial case and calculate n!

EXAMPLE OF RECURRSION - THE TOWER OF HANOI PUZZLE

Move the discs from the first to the third peg with a minimum number of strokes

Example

We have 3 pegs. n disks are arranged on the first peg by size so that it is the largest at the bottom. The task is to move the disks from the first to the thirdpeg following the following rules:

- Only one disc is allowed to be moved in one go

- It is not allowed to put a larger disc over a smaller one

- It is necessary to determine the minimum number of moves required to perform this task

Let Tn denote the number of moves needed to solve a problem with n disks. For the basic case n = 1, one move is required, so T1 = 1.

For n = 2, T2 = 3.

For n disks, first move n - 1 disks to the middlepeg in Tn - 1 move, then move the largest disk to the thirdpeg and finally n - 1 disks from the middlepeg to the thirdpeg in Tn - 1 move
The total number of moves is Tn = 2Tn - 1 + 1
Now we have the solution of the problem in recurrsive form: T1 = 1, Tn = 2Tn - 1+ 1

Solution: Tn = 2^n - 1

Poglavlje 2

RECURRENT RELATIONS

DEFINITION OF A RECURRENCE RELATION

A recurrent relation is an equation in which the n-th member of a sequence is defined over its predecessors

A recurrent relation is an equation in which the n-th member of a sequence is defined over its predecessors.

The recurrent relation for the sequence $\{a_n\}$ is an equation expressed over its previous members of the sequence, which are $a_1, \ldots, a_{n-1}$, for all integers n such that $n >= n_0$, where n_0 is a non-negative integer. A sequence is called a solution of a relation if its members satisfy a recurrent relation.

Equation

$a_k = 2a_{k-1}$, or, equivalent to $a_{k+1} = 2a_k$,
in which one of the elements of the sequence is defined over the previous elements of that same sequence represents a recurrent relation.
A recurrent relation of order k is a function of form

$a_n = \text{phi}\,(a_{n-1}, a_{n-2}, \ldots, a_{n-k})$

where the n-th element an array (a_k) is a function of the preceding k elements $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$
By solving the recurrent equation we mean translating the function a(n), which is defined over non-negative integers, from the form in which it is described by a recursive formula to the closed form, where the function a(n) is expressed by an expression directly dependent on n.
The equation $a_0 = 2$, which gives the concrete value of one of the elements of the array, is called the initial condition.
The function $a_n = 3 * 2^n$, which gives the formula for a_n in the function of n, is called the solution of the recurrent relation.

POSSIBLE SOLUTIONS OF RECURRENT RELATIONS

It can happen that there are many arrays that satisfy a given recurrent relation

It can happen that there are many arrays that satisfy a given recurrent relation. For example, bot of sequence

1, 2, 4, 8, 16, . . .

7, 14, 28, 56, 112, . . .

represent a solution of a recurrent relation

$a_k = 2a_{k-1}$.

All such solutions form the so-called general solution of the recurrent relation.

There is usually a single solution of a recurrent relation that satisfies the given initial condition at the same time. For example, the initial condition $a_0 = 3$, a unique solution is:

3, 6, 12, 24, . . .

recurrent relations

$\mathbf{a_k = 2a_{k-1}}$.

If a_n is a sequence that satisfies the recurrent relation $a_n = a_{n-1} + 3$ for n = 1, 2, 3,. . . , and if $a_0 = 2$, then its members are a_1, a_2, and a_3

$a_1 = a_0 + 3 = 2 + 3 = 5$

$a_2 = 5 + 3 = 8$

$a_3 = 8 + 3 = 11$.

AN EXAMPLE OF A RECURRENT RELATION IN THE CALCULATION OF COMPOUND INTERESET

Suppose that a person deposits $10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Example: Suppose that a person deposits $10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Solution:
To solve this problem, let P_n denote the amount in the account after n years. Because
the amount in the account after n years equals the amount in the account after n−1 years plus
interest for the nth year, we see that the sequence $\{P_n\}$ satisfies the recurrence relation
$\mathbf{P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}}$.

The initial condition is P0 = 10,000.
We can use an iterative approach to find a formula for Pn. Note that
P1 = (1.11)P0
P2 = (1.11)P1 = (1.11)2P0
P3 = (1.11)P2 = (1.11)3P0
⋮
Pn = (1.11)Pn−1 = (1.11)nP0.
When we insert the initial condition P0 = 10,000, the formula Pn = (1.11)n10,000 is obtained.
Inserting n = 30 into the formula Pn = (1.11)n10,000 shows that after 30 years the account
contains
P30 = (1.11)3010,000 = $228,922.97.

HOW CAN WE DETERMINE THE OTHER MEMBERS OF THE SEQUENCE BASED ON THE FIRST MEMBERS 1/2

A common problem in discrete mathematics is finding a closed formula, a recurrence relation

A common problem in discrete mathematics is finding a closed formula, a recurrence relation, or some other type of general rule for constructing the terms of a sequence. Sometimes only a
few terms of a sequence solving a problem are known; the goal is to identify the sequence. Even
though the initial terms of a sequence do not determine the entire sequence (after all, there are
infinitely many different sequences that start with any finite set of initial terms), knowing the first few terms may help you make an educated conjecture about the identity of your sequence.
Once you have made this conjecture, you can try to verify that you have the correct sequence.
When trying to deduce a possible formula, recurrence relation, or some other type of rule for the terms of a sequence when given the initial terms, try to find a pattern in these terms. You might also see whether you can determine how a term might have been produced from those proceding it. There are many questions

you could ask, but some of the more useful are but some og the most useful are:

- Are there runs of the same value? That is, does the same value occur many times in a row?
- Are terms obtained from previous terms by adding the same amount or an amount that depends on the position in the sequence?
- Are terms obtained from previous terms by multiplying by a particular amount?
- Are terms obtained by combining previous terms in a certain way?
- Are there cycles among the terms?

Task:

How can we produce the terms of a sequence if the first 10 terms are 1, 2, 2, 3, 3, 3, 4, 4, 4, 4?

Solution: In this sequence, the integer 1 appears once, the integer 2 appears twice, the integer 3
appears three times, and the integer 4 appears four times. A A reasonable rule for generating this sequence is that the integer n appears exactly n times, so the next five terms of the sequence would all be 5, the following six terms would all be 6, and so on. The sequence generated this way is a possible match.

HOW CAN WE DETERMINE THE OTHER MEMBERS OF THE SEQUENCE BASED ON THE FIRST MEMBERS - 2/2

Exercises

Task

How can we produce the terms of a sequence if the first 10 terms are 5, 11, 17, 23, 29, 35, 41,
47, 53, 59?

Solution:

Note that each of the first 10 terms of this sequence after the first is obtained by adding 6 to the previous term. (We could see this by noticing that the difference between consecutiveterms is 6.) Consequently, the nth term could be produced by starting with 5 and adding 6 a total of $n - 1$ times; that is, a reasonable guess is that the nth term is $5 + 6(n - 1) = 6n - 1$. (This is an arithmetic progression with $a = 5$ and $d = 6$.)

Exercises:

Task

For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms
of the sequence.

a) 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, ...
b) 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, ...
c) 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, ...
d) 3, 6, 12, 24, 48, 96, 192,...
e) 15, 8, 1,−6,−13,−20,−27,...
f) 3, 5, 8, 12, 17, 23, 30, 38, 47,...
g) 2, 16, 54, 128, 250, 432, 686,...

CODEWORD ENUMERATION

The following example shows how a recurrent relation can be used to model the number of code words that can be used without passing certain validation checks

Task

A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid,
whereas 120987045608 is not valid. Let an be the number of valid n-digit codewords. Find a recurrence relation for a_n.

Solution: Note that $a_1 = 9$ because there are 10 one-digit strings, and only one, namely, the string 0, is not valid. A recurrence relation can be derived for this sequence by considering how a valid n-digit string can be obtained from strings of $n - 1$ digits. There are two ways to form a valid string with n digits from a string with one fewer digit.
First, a valid string of n digits can be obtained by appending a valid string of $n - 1$ digits with a digit other than 0. This appending can be done in nine ways. Hence, a valid string with n digits can be formed in this manner in $9a_{n-1}$ ways.
Second, a valid string of n digits can be obtained by appending a 0 to a string of length $n - 1$ that is not valid. (This produces a string with an even number of 0 digits because the invalid string of length $n - 1$ has an odd number of 0 digits.) The number of ways that this can be done equals the number of invalid $(n - 1)$-digit strings. Because there are 10^{n-1} strings of length $n - 1$, and a_{n-1} are valid, there are 10^{n-1} valid n-digit strings obtained by appending an invalid string of length $n - 1$.
Because all valid strings of length n are produced in one of these two ways, it follows that there are

$$a_n = 9a_{n-1} + (10^{n-1} - a_{n-1})$$
$$= 8a_{n-1} + 10^{n-1}$$

Poglavlje 3

A linear homogeneous recurrence relation

RECURRENT RELATION WITH CONSTANT COEFFICIENTS

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form
$a_n = c1a_{n-1} + c2a_{n-2} + \cdots + cka_{n-k} + f(n)$
w
here c1, c2, ... , ck are real numbers, and ck ≠ 0, and f is a function of n.

The use of the term linear comes from the fact that the degrees or products of the elements a j do not occur.
The coefficients of the terms of the sequence are all constants, rather than functions that depend on n. The degree is k because an is expressed in terms of the previous k terms of the sequence.
Obviously we can find the expression for an if we know the values of a_{n-1}, a_{n-2},. . . , a_{n-k}.
Thus, by mathematical induction, there is a unique array that satisfies the recurrent relation if the initial values of the first k elements of the array are given.

Example

Find all solutions of the recurrent relations:

an = 5a n−1 − 4a n−2 + n ^ 2

Solution: This is anonhomogeneous recurrence relation ofsecond order with constant coefficients with constant coefficients becausen^2.

If the initial conditions

$a_1 = 1$ i $a_2 = 2$,

then we can successfully find the next few elements of the array

$a_3 = 5 * 2 - 4 * 1 + 3^2 = 15$

$a_4 = 5 * 15 - 4 * 2 + 4^2 = 83$, ...

EXAMPLES OF RECURRENT RELATIONS WITH CONSTANT COEFFICIENTS

$a_n = c_1 a_{n−1} + c_2 a_{n−2} + ⋯ + c_k a_{n−k} + f(n)

Example1

Consider a recurrentrelation

$a_n = 2a_{n-1}a_{n-2} + n^2$ .M

This recurrent relation is not linear because there is a product

$a_{n-1} * a_{n-2}$.

If the initial conditions are

$a_1 = 1$and $a_2 = 2$,

then we can successfully find the following few elements of the sequence

$a_3 = 2 * 2 * 1 + 3^2 = 13$

$a_4 = 2 * 13 * 2 + 4^2 = 68, \ldots$

Example 2

Consider a recurrent relation

$a_n = na_{n-1} + 3a_{n-2}$

This is a homogeneous linear recurrent relation of the second order but does not have a constant coefficient, because the coefficient with a_{n-1} is equal to 0, which is not a constant.

If the initial conditions are

$a_1 = 1$ and $a_2 = 2$,

then we can successfully find the following few elements of the sequence
$a_3 = 3 * 2+3 * 1=9$
$a_4= 4 * 9+3 * 2=42$

Example3

Consider a recurrent relation

$a_n = 2a_{n-1}+5a_{n-2}- 6a_{n-3}$

This is a homogeneous linear recurrent relation of the third order. Therefore, three initial conditions are needed to obtain a unique solution of the recurrent relation. If the initial conditions $a_1 = 1$, $a_2 = 2$, $a_3=1$, then we can successfully find the following few elements of the sequence

$a_4 = 2 * 1 + 5\ 2 - 6 * 1 = 6$

$a_5 = 2 * 6 + 5 * 1 - 6 * 2 = 5$

$a_6 = 2 * 5 + 5 * 6 - 6 * 1 = 34$

TEST YOUR KNOWLEDGE

Are the following relations linear homogeneous recurrent relations of the second order?

a) $a_k = 3a_{k-1} + 2a_{k-2}$
b) $b_k = b_{k-1} + b_{k-2} + b_{k-3}$
c) $c_k = \frac{1}{2}c_{k-1} - \frac{3}{7}c_{k-2}$
d) $d_k = d_{k-1}^2 + d_{k-1} \cdot d_{k-2}$
e) $e_k = 2e_{k-2}$
f) $f_k = 2f_{k-1} + 1$
g) $g_k = g_{k-1} + g_{k-2}$
h) $h_k = (-1)h_{k-1} + (k-1)h_{k-2}$
a. Yes; A = 3 and B = 2
b. No
c. Yes; A = $\frac{1}{2}$ i B = - $\frac{3}{7}$
d. Not; it is not linear
e. Yes; A = 0 and B = 2
f. Not, it is not homogeneous
g. Yes; A = 1 and B = 1
h. Not; there are no constant coefficients.

Poglavlje 4

HOMOGENEOUS LINEAR RECURRENT RELATIONS OF THE SECO

SECOND ORDER RECURRENT RELATION

The polynomial Δ is called the characteristic polynomial of the recurrent relation, and the solutions of the polynom Δ (x) are called the characteristic roots

Consider a second-order homogeneous recurrent relation with constant coefficients that has the form
$a_n = sa_{n-1} + ta_{n-2}$
or
$a_n - sa_{n-1} - ta_{n-2} = 0$
where s and t are coefficients, and valid $t \neq 0$.

Let us join the next square polynomial to this recurrent relation

$\Delta(x) = x^2 - sx - t$. is called the *characteristic polynomial* of the recurrent relation, and the solutions of the polynomial Δ (x) are called the *characteristic roots*. Let the characteristic polynomial

$\Delta(x) = x^2 - sx - t$ recurrent relations $a_n = sa_{n-1} + ta_{n-2}$**has two different roots (solutions),** r_1 **i** r_2**.**

Then the general solution of the recurrent relation is given with $a_n = c_1r_1^n + c_2r_2^n$

where c_1 i c_2 are arbitrary constants that are determined in a unique way from the initial conditions.

Example
Consider a homogeneous recurrent relation
$a_n = 2a_{n-1} + 3a_{n-2}$
Its characteristic polynomial is
$\Delta(x) = x^2 - 2x - 3 = (x - 3)(x + 1)$
which has roots $r_1 = 3$ and $r_2 = -1$
Since the roots are different, we can apply the previous theorem to obtain a general solution
$a_n = c_1 3^n + c_2 (-1)^n$
with arbitrary constants c_1 and c_2

If the initial conditions are $a_0 = 1$ and $a_1 = 2$, then we get:

For n = 0, $a_0 = 1$: $c_1 3^0 + c_2 (-1)^0 = 1$ or $c_1 + c_2 = 1$

For n = 1, $a_1 = 2$: $c_1 3^1 + c_2 (-1)^1 = 2$ or $3c_1 - c_2 = 2$

By solving the system of linear equations by c_1 and c_2 we get $c_1 = 3/4$ and $c_2 = 1/4$, and, therefore,

$a_n = 3/4 * 3^n + 1/4 * (- 1)^n = (3^{n+1} + (-1)n) / 4$ is a general solution of a recurrent relation with initial conditions $a_0 = 1$ i $a_1 = 2$.

EXAMPLE - REAL AND DIFFERENT POLYNOMIAL ROOTS

Examples of recurrent homogeneous linear recurrent relations with different roots

Example

Find a solution

$a_n = a_{n-1} + 6a_{n-2}$ when $a_0 = 3$, $a_1 = 6$

The characteristic polynomial of this recurrent relation is

$r^2 - r - 6 = 0$

The roots of this characteristic polynomial are . $r_1 = 3$ i $r_2 = -2$.

Since the roots of the characteristic polynomial are different, we look for a solution using a_n

$a_n = C_1 3^n + C_2(-2)$, for C_1 i C_2 .

$a_0 = 3 = C_1 + C_2$

$a_1 = 6 = 3C_1 - 2C_2$

By solving the system of equations we get that $C_1 = 2,4$, $C_2 = 0,6$

General solution is a_n

$a_n = 2,4 \cdot (3^n) + 0,6 \cdot (-2)^n.$

Example

Find a solution

$a_n = 7a_{n-1} - 10a_{n-2}$, kada su $a_0 = 2$, $a_1 = 1$

The characteristic polynomial of this recurrent relation is

$r^2 - 7r + 10 = 0$

The roots of this characteristic polynomial are $r_1 = 2$ i $r_2 = 5$.
Since the roots of the characteristic polynomial are different, we look for a solution using a_n
$a_n = C_1 2^n + C_2 5^n$, for some constants C_1 i C_2 .

Having in mind the given initial conditions, we get the following system of equations

$a_0 = 2 = C_1 + C_2$
$a_1 = 1 = 2C_1 + 5C_2$

By solving the system of equations we get that $C_1 = 3$, $C_2 = -1$
So the general solution for is a_n

$a_n = 3 \cdot 2^n - 5^n.$

EXAMPLE OF A RECURRENT RELATION - FIBONACCI SEQUENCE

The Fibonacci sequence is a homogeneous recurrent relation of the second order with constant coefficients

Consider the following recurrent relation for the Fibonacci sequence
$a_n = a_{n-1} + a_{n-2}$
with initial conditions $a_0 = 0$ i $a_1 = 1$

It is a homogeneous recurrent relation of the second order with constant coefficients. Its characteristic polynomial is

$\Delta(x) = x^2 - x - 1$
General solution
a_n=c_1 * (1 + sqrt{ 5 }/ 2 }) ^{n} + c_2 * (1-sqrt{ 5 } / 2 }) ^{n}
Initial conditions lead us to the following system of two linear equations
For n=0, a_0=0: c_1+c 2
Za n=1, a_1=1:

1= $c_1(\frac{1+\sqrt{5}}{2}) + c_2(\frac{1-\sqrt{5}}{2})$

- Koji ima rešenje

$c_1 = \frac{1}{\sqrt{5}}$

$c_1 = -\frac{1}{\sqrt{5}}$

Prema tome, rešenje Fibonačijeve rekurentne relacije je

$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

EQUAL POLYNOMIAL ROOTS

The general solution of a recurrent relation with equal roots $a_n = c_1 r_0^n + c_2 n r_0^n$

When the roots of a characteristic polynomial are equal, **the following result holds.**
Let be a characteristic polynomial

$\Delta(x) = x^2 - sx - t$

recurrent relation

There is only one root r_0, that is, the roots coincide. Then the general solution is a recurrent relation

$a_n = c_1 r_0^n + c_2 n r_0^n$

where c_1 and c_2 are constants that can be determined in a unique way from the initial conditions..

Example
Consider a homogeneous recurrent relation
$a_n = 6a_{n-1} - 9a_{n-2}$
Its characteristic polynomial
$\Delta(x) = x^2 - 6x + 9 = (x - 3)^2$

It has only one root $r_0 = 3$.

This recurrent relation has a general solution
$a_n = c_1 3^n + c_2 n 3^n$

If the initial conditions $a_1 = 3$ i $a_2 = 27$, then

For $n = 1$, $a_1 = 3 : 3 = c_1 3^1 + c_1 * 1 * 3^1$ or
$3 = 3c_1 + 3c_2$
For $n = 2$, $a_2 = 27 : 27 = c_1 3^2 + c_1 * 2 * 3^2$ or

$27 = 9c_1 + 18c_2$

or, equivalently
$1 = c_1 + c_2$
$9 = 3c_1 + 18c_2$

Solutions are $c_1 = -1$ i $c_2 = 2$
The general solution of a recurrent relation with initial conditions
$a_1 = 3$ i $a_2 = 27$
$a_n = -3^n + 2n3^n = 3^n (2n - 1)$

EXAMPLE - EQUAL POLYNOMIAL ROOTS

Find the solution

Find a solution for

$a_n = 2a_{n-1} - a_{n-2}$, u $a_0 = 4$, $a_1 = 1$

The characteristic polynomial of this recurrent relation is
$r^2 - 2r + 1 = 0$
The roots of this characteristic polynomial are $r_{1,2} = 1$.
Since the roots of the characteristic polynomial are the same, for an we look for a solution using
$a_n = C_1 1^n + C_2 n 1^n = C_1 + C_2 n$, for C_1 i C_2 .

Having in mind the given initial conditions, we get the following system of equations

$a_0 = 4 = C_1 + C_2(0)$
$a_1 = 1 = C_1 + C_2(1)$
By solving the system of equations we get that they are $C_1 = 4$, $C_2 = -3$
So the general solution is a_n

$a_n = 4 - 3n$.

$a_n = 2a_{n-1} - 4a_{n-2}$

$a_n = 2a_{n-1} - 4a_{n-2}$

EXAMPLE 2- EQUAL ROOTS OF A POLYNOMIAL

An example of a recurrent homogeneous linear recurrent relation with the same roots.

Suppose a string $b_0, b_1, b_2, \ldots$ satisfies the recurrent relation
$b_k = 4b_{k-1} - 4b_{k-2}$ for all integers $k \geq 2$

with initial conditionsa $b_0 = 1$ i $b_1 = 3$

Find a unique solution for the string $b_0, b_1, b_2, \ldots$.

We can notice that it is a linear homogeneous recurrent relation of the second order with the same roots of the characteristic polynomial. Solution of a characteristic polynomial
$t^2 - 4t + 4 = 0$

is $r_{1,2} = 2$

$b_n = C_1 \cdot 2^n + C_2 n 2^n$for all integers $n \geq 0$,
where C_1 i C_2 rreal numbers whose value is determined using initial conditions $b_0 = 1$ i $b_1 = 3$. As we would determine C_1 i C_2 we set up a system of equations

$b_0 = 1 = C_1 \cdot 2^0 + C_2 \cdot 0 \cdot 2^0 = C$
$b_1 = 3 = C_1 \cdot 2^1 + C_2 \cdot 1 \cdot 2^1 = 2C_1 + 2C_2$.

EXAMPLE 2- EQUAL ROOTS OF A POLYNOMIAL - 2 / 2

An example of a recurrent homogeneous linear recurrent relation with the same roots. 2/2

By solving the system of equations, we come to the conclusion that it is:

C_1 = 1
and
2 C $_1$ + 2 C _$_2$ = 3.

Substitution C_1 = 1 in the second equation gives

2 + 2 C_2 = 3 , C_2 = 1/2

By substituting the obtained values for C_1 and C_2 we get a general solution of the sequence

b_$_n$ = 2 ^{n} + 1/2 * n * 2 ^{n} = 2 ^{n} (1 + n/2) for all integers n >=0.

Poglavlje 5

EXERCISE - RECURRENT RELATIONS

TASK1

An example of a recurrent relation with two zeros of a characteristic polynomial

Estimated duration: 15 minutes

Find the solution of recurerent ralion an **= 3a $_{n+1}$ − a $_{n+2}$** , sa with unitinoal conditins: a0 **= 0 i a$_1$ = 1.**

Solution:

$$a_{n+2} - 3a_{n+1} + a_n = 0$$

$$x^2 - 3x + 1 = 0$$

$$x_1 = \frac{3+\sqrt{5}}{2},$$

$$x_2 = \frac{3-\sqrt{5}}{2}$$

$$a_n = c_1 x_1^n + c_2 x_2^n$$

$$a_n = c_1 (\frac{3+\sqrt{5}}{2})^n + c_2 (\frac{3-\sqrt{5}}{2})^n$$

$$a_0 = c_1 + c_2 = 0$$

$$a_1 = c_1 \frac{3+\sqrt{5}}{2} + c_2 \frac{3-\sqrt{5}}{2} = 1$$

$$-c_1 \frac{3-\sqrt{5}}{2} - c_2 \frac{3-\sqrt{5}}{2} = 0$$

$$c_1 \frac{3+\sqrt{5}}{2} + c_2 \frac{3-\sqrt{5}}{2} = 1$$

$$c_1 (\frac{3+\sqrt{5}}{2} + \frac{3-\sqrt{5}}{2}) = 1$$

$$c_1 = 1/\sqrt{5}$$

$$1/\sqrt{5} + c_2 = 0$$

$$c_2 = -1/\sqrt{5}$$

$$a_n = 1/\sqrt{5}(\frac{3+\sqrt{5}}{2})^n - 1/\sqrt{5}(\frac{3-\sqrt{5}}{2})^n$$

TASK 2

Recurrent relation solution:

Estimated time: 10 minuta

Find the solution of the recurrent relation:

a_{n+2} = 3a_{n+1} - 2an with **a_0 = 5 i a_1 = 6**

SOLUTION:

$$a_{(n+2)} - 3a_{(n+1)} + 2a_n = 0$$

$$x^2 - 3x + 2 = 0$$

$$x_1 = 1, x_2 = 2$$

$$a_n = c_1 x_1^n + c_2 x_2^n$$

$$a_0 = c_1 1^0 + c_2 2^0$$

$$a_0 = c_1 + c_2 = 5$$

$$a_1 = c_1 x_1^+ c_2 x_2^= 6$$

$$-c_1 - c_2 = -5$$

$$c_1 x_1 + c_2 x_2 = 6$$

$$-c_1 - c_2 = -5$$

$$2c_1 + c_2 = 6$$

$$c_1 = 1 \Longrightarrow c_2 = 4$$

$$a_n = 2^n + 4$$

TASK 3

Recurrent relation solution: $a_{n+2} = 4a_{n+1} - 4a_n$ $a_{n+2} = 4a_{n+1} - 4a_n$

Estimated time: 15 minuta

Define the definition of a recursive sequence an such that:

a_{n+2} = 4a_{n+1} - 4a_n sa početnim uslovima a_1 = 4 i a_2 = 13

Suppose a string $b_0, b_1, b_2, ...$satisfies the recurrent relation

$b_k = 4b_{k-1} - 4b_{k-2}$ for all integers $k \geq 2$

with initial conditionsa $b_0 = 1$ i $b_1 = 3$

Find a unique solution for the string $b_0, b_1, b_2, ...$

M
$t^2 - 4t + 4 = 0$

is
$r_{1,2} = 2$

It follows that we use a statement

$b_n = C_1 \cdot 2^n + C_2 n 2^n$za sve cele brojeve $n \geq 0$,
where C_1 i C_2 realni brojevi čija se vrednost određuje koristeći početne uslove $b_0 = 1$ i $b_1 = 3$.
Kako bi smo odredili C_1 i C_2 postavljamo sistem jednačina

$b_0 = 1 = C_1 \cdot 2^0 + C_2 \cdot 0 \cdot 2^0 = C$
$b_1 = 3 = C_1 \cdot 2^1 + C_2 \cdot 1 \cdot 2^1 = 2C_1 + 2C_2$

TASK 4

Determining the definition of a recursive sequence

Estimated time: 5 minuts

Define the definition of a recursive sequence an such that:

$$a_n = \frac{n!}{(n-1)}$$

za n $\in \mathbb{N} \setminus \{1\}$

Soluition:

$$a_n = \frac{n!}{(n-1)}$$

$$a_n = \frac{(n-1)! \cdot n \cdot (n-2)}{(n-1) \cdot (n-2)}$$

$$a_n = \frac{n \cdot (n-2)}{(n-1)} - a_{n-1}$$

TASK 5

Determine the general solution of the recurrent relation an= 5 a (n-1) - 6 an - 2

Estimated time: 10 minuts

Determine the general solution of the recurrent relation

$a_n = 5\, a_{(n-1)} - 6\, a_{n-2}$

$a_0 = 1$ i $a_1 = 4$.

The solution of the recurrent relation $a_n = 5a_{n-1} - 6a_{n-2}$

Solution:

The characteristic equation is:

$x^2 - 5x + 6 = 0$.

Roots are:

x=2 i x=3,

every solution of the equation has a form

$a_n = \alpha\, 3^n + \beta\, 2^n$.

$a_0 = \alpha + \beta = 1$

$a_1 = 3\alpha + 2\beta = 4$

By solving the system of equations, we get that **$\alpha = 2$ i $\beta = -1$. The solution of the equation with the given initial conditions is:**
$a_n = 2 * 3^n - 2^n$.

TASK 6

Determine the general solution of the recurrent relation: an = 6a n-1- 9an – 2

Estimated time: 10 minuts

Determine the general solution of the recurrent relation:

$a_n = 6a_{n-1} - 9a_{n-2}$

Initial conditions are:
$a_0 = 4$ i $a_1 = 6$.

Solution:

The characteristic equation is

$x^2 - 6x + 9 = (x-3)^2 = 0.$

Since x = 3 is the only real solution, the solution is of the form:

$a_n = \alpha 3^n + \beta n 3^n$

$a_0 = \alpha = 4$

$a_1 = 3\alpha + 3\beta = 6$

By solving the system we get: **$\alpha = 4$ i $\beta = -2$.**

The solution of the equation is:
$a_n = 4 \cdot 3^n - 2n \cdot 3^n$

TASK 7

Solving determinants of order n

Estimated time: 10 minuta

Solving determinants of order n

$$D_n(-5, 10, -5) = \begin{vmatrix} 10 & -5 & 0 & 0 & \dots & 0 & 0 \\ -5 & 10 & -5 & 0 & \dots & 0 & 0 \\ 0 & -5 & 10 & -5 & \dots & 0 & 0 \\ 0 & 0 & -5 & 10 & & 0 & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -5 & 10 & -5 \\ 0 & 0 & 0 & \dots & 0 & -5 & 10 \end{vmatrix}.$$

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Solution:
If we develop a given determinant first by the first row, and then by
in the first column we get a recurrent connection

$D_n = 10D_{n-1} - 25D_{n-2}$

(D_{n-1} i D_{n-2} they represent determinants of the same form but in less rows n − 1 i n − 2).

The corresponding characteristic equation is

$t^2 - 10t + 25 = 0$ and it has a double realistic solution

$t_1 = t_2 = 5$, the general solution is

$Dn = C_1 * 5^n + C_2 * n * 5^n$.

and it has a double realistic solution

$$D_1 = 10$$

$$D_2 = 75$$

$$-----------------------$$

$$5C_1 + 5C_2 = 10$$

$$25C_1 + 50C_2 = 75$$

$$-------------------------$$

$$C_1 = C_2 = 1 \Rightarrow D_n = 5^n \cdot (n+1).$$

TASK 8

Solution of the recurrent relation $a_{n+2} = 2a_{n+1} - 2a_n$

Estimated time: 15 minutes

Find a solution to the recurrent relation

$a_{n+2} = 2a_{n+1}$ - $2a_n$ with initial conditions a0 = 1 i a_1 = 1 + i

SOLUTION:

The characteristic equation is $x^2 - 2x + 2 = 0$. Its solutions are conjugate complex numbers: $x_1 = 1 + i$ i $x_2 = 1 - i$.

> Trigonometric notation of complex number:
> z = x + iy
> z = r(cos θ + i sin θ), r ≥ 0, θ \in {0, 2 \pi }.

Solution:

If we develop a given determinant first by the first row, and then by
in the first column we get a recurrent connection

$D_n = 10D_{n-1} - 25D_{n-2}$

(D_{n-1} i D_{n-2} they represent determinants of the same form only in smaller orders: n − 1 i n − 2).

The general solution of this equation is of the form:

$a_n = C_1 \rho^n \cos(n\theta) + C_2 \rho^n \sin(n\theta)$.

Let us determine the trigonometric solutions:

$$x_1 = \sqrt{2} \cdot (cos\frac{\pi}{4} + i \cdot sin\frac{\pi}{4}), x_2 = \sqrt{2} \cdot (cos\frac{\pi}{4} - i \cdot sin\frac{\pi}{4})$$

Let's look at what the general expression for initial conditions looks like:

$$a_0 = C_1 \cdot \rho^0 \cdot cos(0 \cdot \theta) + C_2 \cdot \rho^0 \cdot sin(0 \cdot \theta)$$

$$a_1 = C_1 \cdot \rho^1 \cdot cos(\theta) + C_2 \cdot \rho^1 \cdot sin(\theta)$$

$$- -$$

$$a_0 = C_1 = 1$$

$$a_1 = \rho \cdot cos(\theta) + C_2 \cdot \rho \cdot sin(\theta) = 1 + i$$

$$C_2 = 1$$

Rešenje jednačine je:

$$a_n = \sqrt{2}^n \cdot cos(n \cdot \frac{\pi}{4}) + \sqrt{2}^n \cdot sin(n \cdot \frac{\pi}{4})$$

Poglavlje 6

Tasks for independent work

TASKS

Exercise tasks

Task 1 - estimated time 10 minutes

Find a solution to the recurrent relation $x_{n+2} = 4x_{n+1} - 4x_n$, with initial conditions $x_1 = 2$, $x_2 = 8$.

Task 2 - estimated time 10 minutes

$a_n = 10a_{n-1} - 25a_{n-2}$ with initial conditions $a_0=3$, $a_1 = 17$

Task 3 - estimated time 10 minutes

Find a solution to the recurrent relation $2x_{n+2} = 6x_{n+1} - 4x_n$, with initial conditions x1 = 2, $x_2 = 8$.

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CONCLUSION

The focus of this lesson was on the notion of recursion. Recurrent relations and cardinality of sets are explained. In this lesson, the notion of infinity, cardinality, recursion is explained. Recurrent relations and their review are presented. As well as application through the above examples.

Literatura

[1] Rosen, Kenneth H. "Discrete mathematics and its applications." AMC 10 (2007): 12.

[2] Epp, Susanna S. Discrete mathematics with applications. Cengage Learning, 2010.